

# A FEW POSTS OF PROBLEMAS | TEOREMAS

AMÉRICO TAVARES

## CONTENTS

1. A note on a combinatorial identity related to the Apéry's constant $\zeta(3)$	1
1.1. Introduction	1
1.2. Proof of the identity	2
1.3. First Terms of Apéry's Sequences	4
1.4. Integer sequence $(u_n)$	4
1.5. Equivalent formulas for $u_n$ and $v_n$	8
2. Putnam problem of the day, HMD, March 1, 2008	10
2.1. English version of the Putnam problem of the day, HMD, March 1, 2008	10
3. Period of a decimal expansion/Período de uma dízima	12
3.1. My solution to the Problem Of the Week-9 [Todd and Vishal's blog]: Period of a decimal expansion	12
4. Congruences and Divisibility	13
4.1. A Purdue University Problem of the Week, Problem No. 12 (Spring 2009 Series)	13
References	14

### 1. A NOTE ON A COMBINATORIAL IDENTITY RELATED TO THE APÉRY'S CONSTANT $\zeta(3)$

**1.1. Introduction.** In his proof of the irrationality of  $\zeta(3)$ , Roger Apéry used two double sequences, whose first terms are shown in section 3. Apéry's method of constructing these sequences is presented in his article [1] *Irrationalité de  $\zeta(2)$  et  $\zeta(3)$* , Astérisque 61 (1979) p. 11-13, and in much more detail by Alfred van der Poorten [2] in his informal report *A proof that Euler missed ...*, Math. Intelligencer 1 (1979), p. 195-203.

In section 4 of [2], the 5 transformations presented by Apéry in the Journées Arithmétiques de Luminy held in June 1978, which enabled him to accelerate the convergence of a sequence,  $u_{n,k}$  are described.

---

*Date:* June 2009.

This sequence is given by

$$u_{n,k} = \sum_{m=1}^n \frac{1}{m^3} + \sum_{m=1}^k \frac{1}{2m^3 \binom{n}{m} \binom{m+n}{m}}. \quad (1)$$

**Remark 1.** *In van der Poorten's paper this sequence is denoted as  $c_{n,k}$  and as  $u_{k,n}$  by Apéry in the aforementioned article.*

Applying exactly the same transformations, that is, the same linear combination to the double sequences  $\binom{n+k}{n}$  and  $\binom{n+k}{n} u_{n,k} = v_{n,k}$ , two other sequences, respectively,  $(u_n)$  and  $(v_n)$  are generated. These are given by

$$u_n = \sum_{i=0}^n \binom{n}{i}^2 \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{k} \quad (2)$$

$$= \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \text{ with } 0 \leq k \leq i \leq n \quad (3)$$

and

$$v_n = \sum_{i=0}^n \binom{n}{i}^2 \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{k} u_{n,n-k} \quad (4)$$

$$= \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 u_{n,k}, \text{ with } 0 \leq k \leq i \leq n \quad (5)$$

We present here the following identity

$$\sum_{i=0}^n \binom{n}{i}^2 \binom{i}{k} = \binom{n}{k} \binom{2n-k}{n}$$

that enables one to show that (2) is identical to (3) and (4) is identical to (5) (see section 4).

**1.2. Proof of the identity.** We are going to prove

$$\sum_{i=k}^n \binom{n}{i}^2 \binom{i}{k} = \binom{n}{k} \binom{2n-k}{n} \quad (6)$$

**Remark 2.** *We can choose for  $i$  the minimum value  $k$ , instead of 0, because, when  $i < k$ ,  $\binom{i}{k} = 0$ .*

This identity generalizes the identity

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

for which two different proves are given by M. Petkovsek, H. Wilf and D. Zeilberger in the book  $A=B$ , p. 24, [5], (a purely combinatorial proof and one based on the Newton's binomial formula).

We will present two purely combinatorial arguments to evaluate the value of each side of identity (6). Both values must, of course, be equal.

The right hand side is the number of different ways of choosing  $k$  elements from a set, such as  $S = \{s_1, s_2, \dots, s_n\}$  with  $n$  elements and, at the same time,  $n$  elements from another set  $X = \{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{2n-k}\}$  with  $2n - k$  elements.

As for the left hand side, let's first consider the disjunction of  $X$  in two sets, one  $X_1 = \{x_1, x_2, \dots, x_n\}$  with  $n$  elements, and the other  $X_2 = \{x_{n+1}, \dots, x_{2n-k}\}$ , with  $n - k$  elements, and  $X = X_1 \cup X_2$ . Now, we choose  $n$  elements of  $X$  such as  $k'$  belongs to  $X_1$  and  $n - k'$  to  $X_2$ , with  $0 \leq k' \leq n$ .

- (1) There exists  $\binom{n}{k'}$  different ways of choosing  $k'$  elements from the  $n$  of  $X_1$ .
- (2) There exists  $\binom{n-k}{n-k'} = \binom{n-k}{k'-k}$  different ways of choosing  $n - k'$  elements from the  $n - k$  of  $X_2$ .

From here we deduce that, for a given  $k'$ , there are  $\binom{n}{k'} \binom{n-k}{k'-k}$  different ways of choosing those  $n$  elements of  $X$ . On the other hand,  $\binom{n}{k'} \binom{n-k}{k'-k} \binom{n}{k}$  is the number of different ways of choosing  $n$  elements from  $X$  and, at the same time,  $k$  from  $S$ , as above. Now, if we add, for every possible value of  $k'$ , all those such chooses, then

$$\begin{aligned} \sum_{k'=0}^n \binom{n}{k'} \binom{n-k}{k'-k} \binom{n}{k} &= \sum_{k'=0}^n \binom{n}{k'}^2 \binom{k'}{k} \\ &= \sum_{i=0}^n \binom{n}{i}^2 \binom{i}{k} \\ &= \sum_{i=k}^n \binom{n}{i}^2 \binom{i}{k}, \end{aligned}$$

since  $\binom{n}{k} \binom{n-k}{k'-k} = \binom{n}{k'} \binom{k'}{k}$ , by the Newton's identity. It can be established by a combinatorial argument (see [6] V.K., Md. Balakrishnan, V. Balakrishnan, *Schaum's Outline of Combinatorics*, p. 12).



$$\begin{array}{cccc}
 & & & 1 \\
 & & & 2 & 3 \\
 & & 6 & 12 & 19 \\
 & 20 & 50 & 92 & 147 \\
 \boxed{4} & \binom{n}{i} \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{n}
 \end{array}$$

$$\begin{array}{cccc}
 & & & 1 \\
 & & & 2 & 3 \\
 & & 6 & 24 & 19 \\
 & 20 & 150 & 276 & 147 \\
 \boxed{5} & \sum_{i=0}^j \binom{j}{i} \binom{n}{i} \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{n}
 \end{array}$$

$$\begin{array}{cccc}
 & & & 1 \\
 & & & 2 & 5 \\
 & & 6 & 30 & 73 \\
 & 20 & 170 & 596 & 1445
 \end{array}$$

From here, we get the first 4 terms of  $(u_n)$

$$u_0 = 1 \quad u_1 = 5 \quad u_2 = 73 \quad u_3 = 1445 \quad (7)$$

1.4.1. *Rational sequence*  $(v_n)$ . The first triangle for this sequence has as elements the values generated by  $\binom{n+k}{n} u_{n,k} = v_{n,k}$ . For the  $n$ -th line, these values are obtained for  $k = 0$  to  $n$ .

$$\begin{array}{ccccccc}
 & & & & 0 & & & & \\
 & & & & 1 & & \frac{5}{2} & & \\
 & & & \frac{9}{8} & \frac{29}{8} & & \frac{115}{16} & & \\
 & \frac{251}{216} & & \frac{130}{27} & \frac{5191}{432} & & \frac{5195}{216} & & 
 \end{array}$$

It is transformed into the following ones, by applying exactly the same transformations; again,  $n$  generates a line, for which the second index is comprised between 0 and  $n$ .

$$\boxed{1} \text{ Substitution of } k \text{ by } n - k: \binom{2n-k}{n} u_{n,n-k}$$

$$\begin{array}{ccccccc}
 & & & & 0 & & & & \\
 & & & & \frac{5}{2} & & 1 & & \\
 & & & \frac{115}{16} & \frac{29}{8} & & \frac{9}{8} & & \\
 & \frac{5195}{216} & & \frac{5191}{27} & \frac{130}{432} & & \frac{251}{216} & & \\
 \boxed{2} & \binom{n}{k} \binom{2n-k}{n} u_{n,n-k}
 \end{array}$$

$$\begin{array}{ccccccc}
 & & & & 0 & & & & \\
 & & & & \frac{5}{2} & & 1 & & \\
 & & & \frac{115}{16} & \frac{29}{4} & & \frac{9}{8} & & \\
 & \frac{5195}{216} & & \frac{5191}{144} & \frac{130}{9} & & \frac{251}{216} & & 
 \end{array}$$

$$\boxed{3} \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{n} u_{n,n-k}$$

$$0$$

$$\frac{10390}{432} \quad \frac{115}{16} \quad \frac{25963}{432} \quad \frac{231}{16} \quad \frac{47776}{432} \quad \frac{365}{16} \quad \frac{76331}{432}$$

**Remark 3.** The denominators of the rationals calculated so far are equal to  $2d_n^3$ , where  $d_n = \text{lcm}(1, 2, \dots, n)$ .

$$\boxed{4} \binom{n}{i} \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{n} u_{n,n-k}$$

$$0$$

$$\frac{10390}{432} \quad \frac{115}{16} \quad \frac{77889}{432} \quad \frac{462}{16} \quad \frac{143328}{432} \quad \frac{365}{16} \quad \frac{76331}{432}$$

$$\boxed{5} \sum_{i=0}^j \binom{j}{i} \binom{n}{i} \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{n} u_{n,n-k}$$

$$0$$

$$\frac{10390}{432} = \frac{5195}{216} \quad \frac{115}{16} \quad \frac{88279}{432} \quad \frac{577}{16} \quad \frac{2673}{432} \quad \frac{12}{2} = 6 \quad \frac{1404}{16} = \frac{351}{4} \quad \frac{62531}{36}$$

From here, we get the first 4 terms of  $(v_n)$

$$v_0 = 0 \quad v_1 = 6 \quad v_2 = \frac{351}{4} \quad v_3 = \frac{62531}{36} \quad (8)$$

Let's see how fast the convergence to  $\zeta(3)$  of the sequence generated by  $\frac{v_n}{u_n}$  really is

$$\frac{v_0}{u_0} = 0 \quad \frac{v_1}{u_1} = \frac{6}{5} = 1.2$$

$$\frac{v_2}{u_2} = \frac{351}{292} \approx 1.20205479452055$$

$$\frac{v_3}{u_3} = \frac{62531}{52020} \approx 1.20205690119185$$

For this last approximation, we have already

$$\left| \frac{v_3}{u_3} - \zeta(3) \right| \approx 0.00000000196774$$

We have taken

$$\zeta(3) \approx 1.20205690315959$$

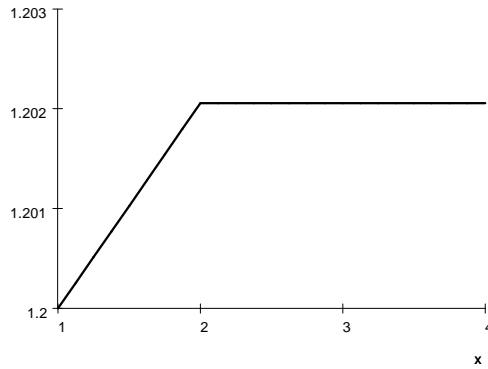
For the next term, we have

$$\frac{v_4}{u_4} = \frac{11424695}{9504288},$$

and

$$\left| \frac{v_4}{u_4} - \zeta(3) \right| \approx 0.000000000000177.$$

In order to have a visual representation of these approximations, let's plot the graph of a function  $f$  of a real variable  $x$  such that  $f(n) = v_n/u_n$  and  $f(x) = (v_n/u_n - v_{n-1}/u_{n-1})(x - n) + v_n/u_n$  in the interval  $[n - 1, n]$ , which is simply equal to  $v_n/u_n$  at the integers and to the linear function of the line segment connecting  $(n - 1, v_{n-1}/u_{n-1})$  to  $(n, v_n/u_n)$  in each of such intervals .



Graph of  $f(x)$  in  $[1, 4]$  approaching  $\zeta(3)$

These approximations compare favorably with the partial sums  $s_n = u_{n,0} = \sum_{m=1}^n 1/m^3$

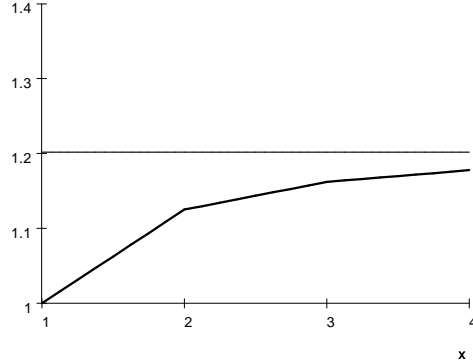
$$s_1 = u_{1,0} = 1$$

$$s_2 = u_{2,0} = 1 + \frac{1}{2^3} = \frac{9}{8}$$

$$s_3 = u_{3,0} = 1 + \frac{1}{2^3} + \frac{1}{3^3} = \frac{251}{216}$$

$$s_4 = u_{4,0} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} = \frac{2035}{1728}$$

In this case we plot the graph of a function  $g$  of a real variable  $x$  such that  $g(n) = s_n$  and  $g(x) = (s_n - s_{n-1})(x - n) + s_n$  in the interval  $[n - 1, n]$ , which is equal to  $s_n$  at the integers and to the linear function of the line segment connecting  $(n - 1, s_{n-1})$  to  $(n, s_n)$  in each of such intervals .

Graph of  $g(x)$  in  $[1, 4]$  and  $\zeta(3)$ 

1.4.2. *Double Sequence*  $(u_{n,k})$ . We will now arrange sequence (1) in a Pascal-like triangle:

$$\begin{array}{cccccccc}
 & & & & u_{0,0} = 0 & & & & \\
 & & & & u_{1,0} = 1 & & u_{1,1} = \frac{5}{4} & & \\
 & & u_{2,0} = \frac{9}{8} & & u_{2,1} = \frac{29}{24} & & u_{2,2} = \frac{115}{96} & & \\
 u_{3,0} = \frac{251}{216} & & u_{3,1} = \frac{65}{54} & & u_{3,2} = \frac{5191}{4320} & & u_{3,3} = \frac{5195}{4320} & & 
 \end{array}$$

For instance, in the 4th line we have the following inequalities

$$u_{3,0} < u_{3,2} < \zeta(3) < u_{3,3} < u_{3,1}$$

that is,  $[u_{3,2}, u_{3,3}] \subset [u_{3,0}, u_{3,1}]$  and  $\zeta(3) - u_{3,0}$  is positive,  $\zeta(3) - u_{3,1}$  negative,  $\zeta(3) - u_{3,2}$  positive, and  $\zeta(3) - u_{3,3}$  negative. Similar results can be established for the other lines.

1.4.3. *Proved identity*. The first 5 lines of

$$\sum_{i=0}^n \binom{n}{i}^2 \binom{i}{k} = \binom{n}{k} \binom{2n-k}{n} \quad (9)$$

are

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 2 & 1 \\
 & & & & & 6 & 6 & 1 \\
 & & & & & 20 & 30 & 12 & 1 \\
 & & & & & 70 & 140 & 90 & 20 & 1
 \end{array}$$

1.5. **Equivalent formulas for  $u_n$  and  $v_n$** . Now we will derive formulas (3) and (5) from, respectively, (2) and (4).

1.5.1. *Formula for  $u_n$ .* From (2), that we repeat here

$$u_n = \sum_{i=0}^n \binom{n}{i}^2 \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{k} \quad (10)$$

one has, using the proved identity

$$u_n = \sum_{i=0}^n \sum_{k=0}^i \binom{n}{i}^2 \binom{i}{k} \binom{n}{k} \binom{2n-k}{k} \quad (11)$$

$$\begin{aligned} &= \sum_{k=0}^i \binom{n}{k} \binom{2n-k}{k} \sum_{i=0}^n \binom{n}{i}^2 \binom{i}{k} \\ &= \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{k} \binom{n}{k} \binom{2n-k}{k} \\ &= \sum_{k=0}^n \binom{n}{k}^2 \binom{2n-k}{n}^2 \end{aligned} \quad (12)$$

Replacing  $n-k$  for  $k$ , we obtain

$$u_n = \sum_{n-k=0}^{n-k=0} \binom{n}{n-k}^2 \binom{2n-n+k}{n}^2 \quad (13)$$

$$\begin{aligned} &= \sum_{k=n}^{k=0} \binom{n}{k}^2 \binom{n+k}{n}^2 \\ &= \sum_{k=0}^{k=n} \binom{n}{k}^2 \binom{n+k}{n}^2 \end{aligned} \quad (14)$$

1.5.2. *Formula for  $v_n$ .* From (4), that we repeat here

$$v_n = \sum_{i=0}^n \binom{n}{i}^2 \sum_{k=0}^i \binom{i}{k} \binom{n}{k} \binom{2n-k}{k} u_{n,n-k} \quad (15)$$

one gets, using the proved identity as well,

$$\begin{aligned} v_n &= \sum_{i=0}^n \sum_{k=0}^i \binom{n}{i}^2 \binom{i}{k} \binom{n}{k} \binom{2n-k}{k} u_{n,n-k} \\ &= \sum_{k=0}^i \binom{n}{k} \binom{2n-k}{k} \sum_{i=0}^n \binom{n}{i}^2 \binom{i}{k} u_{n,n-k} \\ &= \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{k} \binom{n}{k} \binom{2n-k}{k} u_{n,n-k} \\ &= \sum_{k=0}^n \binom{n}{k}^2 \binom{2n-k}{n}^2 u_{n,n-k} \end{aligned} \quad (16)$$

Replacing  $n - k$  for  $k$ , yields

$$\begin{aligned}
 v_n &= \sum_{n-k=0}^{n-k=n} \binom{n}{n-k}^2 \binom{2n-n+k}{n}^2 u_{n,k} \\
 &= \sum_{k=n}^{k=0} \binom{n}{k}^2 \binom{n+k}{n}^2 u_{n,k} \\
 &= \sum_{k=0}^{k=n} \binom{n}{k}^2 \binom{n+k}{n}^2 u_{n,k}
 \end{aligned} \tag{17}$$

This completes the deduction we have proposed to.  $\square$

## 2. PUTNAM PROBLEM OF THE DAY, HMD, MARCH 1, 2008

### 2.1. English version of the Putnam problem of the day, HMD, March 1, 2008.

On March 1, 2008, the Putnam problem of the day displayed on the Harvard's Math Department site

[<http://www.math.harvard.edu/putnam/index.html>]

was stated as follows:

"Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}} \tag{18}$$

Express your answer in the form  $\frac{a + b\sqrt{c}}{d}$ , where  $a, b, c, d$  are integers."

#### **Solution:**

To evaluate the radicand I start by seeing that the continued fraction

$$x = \frac{1}{2207 - \frac{1}{2207 - \dots}}$$

satisfies

$$x = \frac{1}{2207 - x}.$$

Thus, since  $\frac{1}{2} \left( 2207 + \sqrt{2207^2 - 4} \right) \approx 2207$ , the only solution left is

$$x = \frac{2207 - \sqrt{2207^2 - 4}}{2}.$$

A few algebraic manipulations give

$$2207 - x = \frac{2207 + 987\sqrt{5}}{2};$$

hence

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}} = \sqrt[8]{\frac{2207 + 987\sqrt{5}}{2}}.$$

In order to have

$$\sqrt[8]{\frac{2207 + 987\sqrt{5}}{2}} = \frac{a + b\sqrt{c}}{d}$$

or equivalently,

$$\frac{d^8}{2} (2207 + 987\sqrt{5}) = (a + b\sqrt{c})^8,$$

with  $a, b, c$  integers,  $d^8/2$  should also be an integer; therefore  $d$  should be even. I assume that  $d = 2$ ; On the other hand  $c$  should be 5. Thus,

$$2^7 (2207 + 987\sqrt{5}) = 126\,336\sqrt{5} + 282\,496 = (a + b\sqrt{5})^8$$

$$\begin{aligned} a + b\sqrt{5} &= \sqrt[8]{2^7 (2207 + 987\sqrt{5})} \\ a &= \sqrt[8]{2^7 (2207 + 987\sqrt{5})} - b\sqrt{5}. \end{aligned}$$

Since, for  $b = 2$

$$a = \sqrt[8]{2^7 (2207 + 987\sqrt{5})} - 2\sqrt{5} < 1,$$

this possibility is excluded. It remains  $b = 1$

$$a = \sqrt[8]{2^7 (2207 + 987\sqrt{5})} - \sqrt{5} \approx 5.2361 - 2.2361 = 3$$

Now I confirm

$$(3 + \sqrt{5})^8 = 126\,336\sqrt{5} + 282\,496.$$

So, the solution I found was

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}} = \frac{3 + \sqrt{5}}{2}. \quad (19)$$

**Remark:** the calculation of

$$(3 + \sqrt{5})^8 = 126\,336\sqrt{5} + 282\,496$$

can be done by hand as follows

$$(3 + \sqrt{5})^2 = 6\sqrt{5} + 14$$

$$(3 + \sqrt{5})^4 = (6\sqrt{5} + 14)^2 = 168\sqrt{5} + 376$$

$$(3 + \sqrt{5})^8 = (168\sqrt{5} + 376)^2 = 126\,336\sqrt{5} + 282\,496.$$

### 3. PERIOD OF A DECIMAL EXPANSION/PERÍODO DE UMA DÍZIMA

**3.1. My solution to the Problem Of the Week-9 [Todd and Vishal's blog]: Period of a decimal expansion.** Find the length of the period of the repeating decimal representation of

$$\frac{1}{65537}.$$

<http://topologicalmusings.wordpress.com/2008/08/23/pow-9-period-of-a-decimal-expansion/>

**My Solution:**

The repeating decimal representation of the number  $1/65537$  is

$$\frac{1}{65537} = 0.\overline{000\,015\,258\,556\dots cba}.$$

Let  $p$  be a prime number. The period of the repeating decimal of  $1/p$  is equal to the order of  $10 \pmod{p}$  and is either  $p - 1$  or a divisor of  $p - 1$ . Since  $65537$  is a prime number, the period of the repeating decimal of  $1/65537$  is either  $65536$  or a divisor of  $65536 = 2^{16}$ . These divisors are

$$k = 2^0, 2^1, 2^2, 2^3, \dots, 2^{16}.$$

By the definition of the order of  $10 \pmod{65537}$ , I have to find the smallest of these  $k = 2^m$  such that

$$10^k \equiv 1 \pmod{65537},$$

which means  $(10^{(2^m)} - 1) / 65537$  should be an integer.

Since

$$10 - 1 < 10^2 - 1 < 10^4 - 1 < 65537.$$

the remaining cases are  $m = 3, 4, \dots, 16$ . From these I have checked in PARI that only

$$\frac{10^{65536} - 1}{65537} = 669179\dots 526527$$

is an integer. For instance

$$\frac{10^{16} - 1}{65537} = \frac{9999999999999999}{65537} \notin \mathbb{Z}.$$

Conclusion: the length of the period of the repeating decimal representation of  $\frac{1}{65537}$  is 65536.

#### 4. CONGRUENCES AND DIVISIBILITY

**4.1. A Purdue University Problem of the Week, Problem No. 12 (Spring 2009 Series). "Problem:** For how many positive integers  $x \leq 10,000$  is  $2^x - x^2$  not divisible by 7?

Justify your answer without the use of computers."

<http://www.math.purdue.edu/pow/spring2009/pdf/problem12.pdf>

Here is my *solution* (accepted).

If  $a \equiv b \pmod{m}$ , then  $a^n \equiv b^n \pmod{m}$ . Applied to  $2^n$  this property gives in general for  $n = 3k + s, 1 \leq s \leq 3, 0 \leq k$

$$2^n \equiv 2^s \pmod{7}, \tag{*}$$

which means that the remainders of the division of  $2^n$  by 7 form a periodic sequence of length 3 starting at  $n = 1$

$$\overbrace{2, 4, 1}^{\text{period}}, \overbrace{2, 4, 1}^{\text{3 terms}}, \dots$$

As for  $n^2$  since (a) if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a + c \equiv b + d \pmod{m}$  and (b) if  $a \equiv b \pmod{m}$ , then  $a^2 \equiv b^2 \pmod{m}$ , we have in general for  $n = 7j + r, 1 \leq r \leq 7, 0 \leq j$

$$n^2 \equiv r^2 \pmod{7} \tag{**}$$

which means that the remainders of the division of  $n^2$  by 7 form a periodic sequence of length 7 starting at  $n = 1$

$$\overbrace{1, 4, 2, 2, 4, 1, 0}^{\text{period}}, \overbrace{1, 4, 2, 2, 4, 1, 0}^{\text{7 terms}}, \dots$$

If  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a - c \equiv b - d \pmod{m}$ . Let  $u_n = 2^n - n^2$ . Therefore from (\*) and (\*\*) we have

$$u_n \equiv 2^s - r^2 \pmod{7}. \tag{***}$$

The remainders of the division of  $u_n$  by 7 form another periodic sequence of length  $21 = \text{lcm}(3, 7)$  which starts also at  $n = 1$ . Four examples of the evaluation of these remainders are presented below.

For  $1 \leq n \leq 21$  the following 15 terms are not divisible by 7:

$$u_1, u_3, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{18}, u_{19}, u_{20}, u_{21}.$$

Hence for  $1 \leq n \leq 9996 = 21 \times \lfloor \frac{10000}{21} \rfloor$ , there are  $15 \times \lfloor \frac{10000}{21} \rfloor = 7140$  terms that are not divisible by 7.

From the remaining 4 terms  $u_{9997}$  and  $u_{9999}$  are not divisible by 7, which gives a total of  $7140 + 2 = 7142$  numbers  $u_n = 2^n - n^2$  not divisible by 7.

Four examples of the evaluation of the remainders:

$$u_9 = 2^9 - 9^2 \quad (9 = 3 \times 2 + 3, s = 3, 9 = 7 \times 1 + 2, r = 2)$$

$$2^9 \equiv 2^3 \pmod{7} \equiv 1 \pmod{7}$$

$$9^2 \equiv 2^2 \pmod{7} \equiv 4 \pmod{7}$$

$$u_9 \equiv 2^3 - 2^2 \pmod{7} \equiv -3 \pmod{7}$$

$$u_{10} = 2^{10} - 10^2 \quad (10 = 3 \times 3 + 1, s = 1, 10 = 7 \times 1 + 3, r = 3)$$

$$2^{10} \equiv 2^1 \pmod{7} \equiv 2 \pmod{7}$$

$$10^2 \equiv 3^2 \pmod{7} \equiv 2 \pmod{7}$$

$$u_{10} \equiv 2^1 - 3^2 \pmod{7} \equiv 0 \pmod{7}$$

$$u_{9997} = 2^{9997} - 9997^2 \quad (9997 = 3 \times 3332 + 1, s = 1, 9997 = 7 \times 1428 + 1, r = 1)$$

$$2^{9997} \equiv 2^1 \pmod{7} \equiv 2 \pmod{7}$$

$$9997^2 \equiv 1^2 \pmod{7} \equiv 1 \pmod{7}$$

$$u_{9997} \equiv 2^{9997} - 9997^2 \pmod{7} \equiv 1 \pmod{7}$$

$$u_{9998} = 2^{9998} - 9998^2 \quad (9998 = 3 \times 3332 + 2, s = 2, 9997 = 7 \times 1428 + 2, r = 2)$$

## REFERENCES

- [1] R. Apéry, *Irrationalité de  $\zeta(2)$  et  $\zeta(3)$* , Astérisque **61** 1979, 11-13.
- [2] A. van der Poorten, *A Proof that Euler Missed...*, Apéry's proof of the irrationality of  $\zeta(3)$ . *An informal report*, Math. Intelligencer **1**, n° 4, 1978/79, pp. 195-203. [http://www.ift.uni.wroc.pl/~mwolf/Poorten\\_MI\\_195\\_0.pdf](http://www.ift.uni.wroc.pl/~mwolf/Poorten_MI_195_0.pdf)  
<http://ega-math.narod.ru/Apery1.htm>
- [3] F. Beukers, *A note on the irrationality of  $\zeta(2)$  and  $\zeta(3)$* , Bull. Lond. Math. Soc. **11**, n° 33, 1978, pp. 268-272
- [4] F. Apéry, *Roger Apéry, 1916-1994: A Radical Mathematician*, Math. Intelligencer **18**, n° 2, 1996, pp. 54-61.
- [5] M. Petkovsek, H. Wilf and D. Zeilberger, *A=B*, 1997.  
<http://www.cis.upenn.edu/~wilf/AeqB.htm>.
- [6] V.K., Md. Balakrishnan, V. Balakrishnan, *Combinatorics*, Schaum's Outline of Combinatorics.

QUELUZ, PORTUGAL

*Current address:* Américo Tavares, Queluz, Portugal

*E-mail address:* [actavares@sapo.pt](mailto:actavares@sapo.pt)

*URL:* <http://www.problemasteoremas.wordpress.com>