problemas | teoremas*
PdM Problema do mês
PoM Problem of the month

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Contents

*http://problemasteoremas.wordpress.com

1 Enunciados :: Statments

1.1 PdM :: PoM #1

1.1.1 Enunciado do Problema

Seja m o maior inteiro positivo tal que $\frac{1}{13}\binom{13^5}{3^7} \in \mathbb{N}$. Determine, justificando,

um majorante de m.

- Nota: não se permite a utilização de calculadoras ou computadores.
- Sairá vencedora a melhor estimativa justificada.
- Afirmação não demonstrada: 10 é um majorante de m. Encontre um mais pequeno.
- O prazo limite para apresentar resoluções é 19.07.2009.

1.1.2 Problem Statement

Let m be the greatest positive integer such that $\frac{1}{13}\binom{13^5}{3^7} \in \mathbb{N}$. Find, with proof, an upper bound for m.

- Remark: the use of calculators or computers is not allowed.
- The best justified estimate will win.
- Claim: 10 is an upper bound for m. Find a smaller one.
- The deadline for submitting solutions is July 19, 2009.

1.2 PdM :: PoM #2

1.2.1 Enunciado do Problema

Admita que $n = 1, 2, 3, \ldots$. Seja $x \geq 0$ um número real, $\binom{x}{0} = 1$ e

$$\binom{x}{n} = \frac{x\,(x-1)\cdots(x-n+1)}{n!}.$$

Deduza a identidade

$$\binom{x}{n} + \binom{x}{n-1} = \binom{x+1}{n}.$$

- O prazo limite para apresentação das resoluções é 9.09.2009, quer via email acltavares@sapo.pt ou comentando no blogue.

1.2.2 Problem Statement

Suppose that $n = 1, 2, 3, \ldots$. Let $x \geq 0$ be a real number, $\binom{x}{0} = 1$ and

$$\binom{x}{n} = \frac{x\,(x-1)\cdots(x-n+1)}{n!}.$$

Derive the identity

$$\binom{x}{n} + \binom{x}{n-1} = \binom{x+1}{n}.$$

- The deadline for submitting solutions is September 9, 2009 either via e-mail acltavares@sapo.pt or comment box.

2 Resoluções :: Solutions

2.1 PdM :: PoM #1 (valoração p-ádica, p-adic valuation)

2.1.1 Solution par Pierre Bernard, France

On sait que

$$v_p\left(\binom{n}{k}\right) = \sum_{i=1}^{+\infty}\left(\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor\right)$$

De plus, chaque terme $\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor$ vaut 0 ou 1 (on a toujours $\lfloor x+y\rfloor - \lfloor x\rfloor - \lfloor y\rfloor$ qui vaut 0 ou 1).

Si i est assez grand, il est clair que $\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor = 0$. Précisément, puisque $n \geq k$, il suffit que $p^i > n$, c'est-à-dire $i > \log_p(n)$ pour que $\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor = 0$. On a donc:

$$v_p\left(\binom{n}{k}\right) = \sum_{i=1}^{\lfloor\log_p(n)\rfloor}\underbrace{\left(\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor\right)}_{0 \text{ ou } 1} \leq \lfloor\log_p(n)\rfloor$$

Donc $v_p\left(\binom{13^5}{3^7}\right) \leq \lfloor\log_{13}(13^5)\rfloor = 5$. Et 5 c'est mieux que 10.

2.1.2 Solution by Pierre Bernard, France; translated by Américo Tavares

We know that

$$v_p\left(\binom{n}{k}\right) = \sum_{i=1}^{+\infty}\left(\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor\right).$$

Furthermore, each term $\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor$ is 0 or 1 (we have always $\lfloor x+y\rfloor - \lfloor x\rfloor - \lfloor y\rfloor$ which is 0 or 1).

For i sufficiently large it is clear that we have $\left\lfloor\frac{n}{p^i}\right\rfloor - \left\lfloor\frac{k}{p^i}\right\rfloor - \left\lfloor\frac{n-k}{p^i}\right\rfloor = 0$. And because $n \geq k$ it is sufficient that $p^i > n$, i. e. $i > \log_p(n)$ to have

$\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor = 0$. Therefore:

$$v_p\left(\binom{n}{k}\right) = \sum_{i=1}^{\lfloor \log_p(n) \rfloor} \underbrace{\left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor\right)}_{0 \text{ or } 1} \leq \lfloor \log_p(n) \rfloor$$

Thus $v_p\left(\binom{13^5}{3^7}\right) \leq \lfloor \log_{13}(13^5) \rfloor = 5$. And 5 is better than 10.

Other solvers: fede (comments in Gaussianos's blog) and fatima.

2.1.3 Resolução de Pierre Bernard, França; tradução de Américo Tavares

Sabe-se que

$$v_p\left(\binom{n}{k}\right) = \sum_{i=1}^{+\infty} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor\right).$$

Além disso, cada termo $\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor$ vale 0 ou 1 (tem-se sempre $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor$ que é igual a 0 ou 1).

Para i suficientemente grande é claro que se tem $\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor = 0$. Ora, dado que $n \geq k$, é suficiente que $p^i > n$, isto é $i > \log_p(n)$ para se ter $\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor = 0$. Portanto:

$$v_p\left(\binom{n}{k}\right) = \sum_{i=1}^{\lfloor \log_p(n) \rfloor} \underbrace{\left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor\right)}_{0 \text{ or } 1} \leq \lfloor \log_p(n) \rfloor$$

Deste modo $v_p\left(\binom{13^5}{3^7}\right) \leq \lfloor \log_{13}(13^5) \rfloor = 5$. E 5 é melhor do que 10.

Outros: fede (commentários no blogue Gaussianos) e fatima.

Notas:

1. $v_p(r)$ designa a valoração (ou valorização) p-ádica (valuation p-adique) de r: o expoente do número primo p na decomposição em factores primos do inteiro r. Por outras palavras, $p^{v_p(r)}$ divide r mas $p^{1+v_p(r)}$ não divide r.

2. Também se usa a notação $\mathrm{ord}_p(r)$ (ordem ou ordinal de r em p) com o mesmo significado.

3. $v_p(\frac{r}{s}) = v_p(r) - v_p(s)$ (com $r \in \mathbb{Q}$).

4. Teorema de Legendre: Qualquer que seja o inteiro positivo n, o expoente do número primo p na decomposição em números primos de $n!$ é igual a

$$\sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor$$

Remarks:

1. $v_p(r)$ denotes the p-adic valuation of r: the exponent of the prime p in the factorization into prime numbers of the integer r. In other words $p^{v_p(r)}$ divides r and $p^{1+v_p(r)}$ does not divide r.

2. With the same meaning another notation is also used: $\mathrm{ord}_p(r)$ (order or ordinal of r at p).

3. $v_p(\frac{r}{s}) = v_p(r) - v_p(s)$ (with $r \in \mathbb{Q}$).

4. Theorem (Legendre): For every positive integer n, the exponent of the prime number p in the factorization into prime numbers of $n!$ is

$$\sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor .$$

2.2 PdM :: PoM #2 (coeficientes da série binomial, binomial series coefficients)

2.2.1 Resolução de antonio girao

[Usou a notação $\dfrac{x!}{(x-n)!} = x\,(x-1)\cdots(x-n+1)$].

$$\binom{x}{n} = \frac{x!}{(x-n)!n!}$$

$$\binom{x}{n-1} = \frac{x!}{(x-n+1)!\,(n-1)!}$$

$$\begin{aligned}
\frac{x!}{(x-n)!n!}+\frac{x!}{(x-n+1)!\,(n-1)!} &= \frac{x!}{(x-n)!n!}+\frac{x!n}{(x-n+1)!\,(n-1)!n} \\
&= \frac{x!\,(x-n+1)}{(x-n)!n!\,(x-n+1)}+\frac{x!n}{(x-n+1)!\,(n-1)!n} \\
&= \frac{x!\,(x-n+1)}{(x-n)!n!\,(x-n+1)}+\frac{x!n}{(x-n+1)!n!} \\
&= \frac{x!\,[(x-n+1)+n]}{(x-n+1)!n!} \\
&= \frac{x!\,(x+1)}{(x-n+1)!n!} \\
&= \frac{(x+1)!}{(x-n+1)!n!} \\
&= \frac{(x+1)!}{(x+1-n)!n!} \\
&= \binom{x+1}{n}
\end{aligned}$$

Outros: Pierre Bernard e MathOMan

2.2.2 Solution by antonio girao

[He used the notation $\dfrac{x!}{(x-n)!} = x\,(x-1)\cdots(x-n+1)$].

$$\binom{x}{n} = \frac{x!}{(x-n)!n!}$$

$$\binom{x}{n-1} = \frac{x!}{(x-n+1)!\,(n-1)!}$$

$$
\begin{aligned}
\frac{x!}{(x-n)!n!}+\frac{x!}{(x-n+1)!\,(n-1)!} &= \frac{x!}{(x-n)!n!}+\frac{x!n}{(x-n+1)!\,(n-1)!n} \\
&= \frac{x!\,(x-n+1)}{(x-n)!n!\,(x-n+1)}+\frac{x!n}{(x-n+1)!\,(n-1)!n} \\
&= \frac{x!\,(x-n+1)}{(x-n)!n!\,(x-n+1)}+\frac{x!n}{(x-n+1)!n!} \\
&= \frac{x!\,[(x-n+1)+n]}{(x-n+1)!n!} \\
&= \frac{x!\,(x+1)}{(x-n+1)!n!} \\
&= \frac{(x+1)!}{(x-n+1)!n!} \\
&= \frac{(x+1)!}{(x+1-n)!n!} \\
&= \binom{x+1}{n}
\end{aligned}
$$

Other solvers: Pierre Bernard and MathOMan

Notas:
1. Estes coeficientes são os da série binomial

$$(1+t)^n = \sum_{n=0}^{\infty} \binom{x}{n} t^n$$

que é convergente para $|t| < 1$.
2. O coeficiente de ordem n é um polinómio de grau n em x.

Remarks:
1. These coefficients are the binomial series ones

$$(1+t)^n = \sum_{n=0}^{\infty} \binom{x}{n} t^n,$$

which is convergent for $|t| < 1$.
2. The coefficient of order n is a polynomial of degree n in x.