#Desafios SPM

#Números Complexos

Exercício 17 [1]
Nível de Complexidade: 3

Seja $\theta \in \mathbb{R}$.

a. Justifique que
$$\cos(\theta) = \frac{\text{cis}(\theta) + \text{cis}(-\theta)}{2}.$$

b. Usando o binómio de Newton, mostre a seguinte fórmula de trigonometria:
$$\cos^6(\theta) = \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}.$$

c. Inspirando-se nas alíneas anteriores, mostre que
$$\sin^4(\theta) = \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}$$
e deduza que $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt[4]{7-4\sqrt{3}}}{2}$.

#propriedades da função cis #fórmulas de Moivre #aplicações à trigonometria #binómio de Newton

Resolução [2]

Nota: uso a notação inglesa $\sin(\cdot)$ para designar a função seno, em vez da portuguesa $\text{sen}(\cdot)$.

a. Pela definição de
$$\text{cis}(\theta) = \cos(\theta) + i\sin(\theta),$$
tem-se que
$$\text{cis}(-\theta) = \cos(-\theta) + i\sin(-\theta)$$
Como $\cos(-\theta) = \cos(\theta)$ e $\sin(-\theta) = -\sin(\theta)$, então
$$\text{cis}(-\theta) = \cos(\theta) - i\sin(\theta)$$

[1] https://static.publico.pt/DOCS/educacao/desafiosSPM_final.pdf?v=2
[2] Data: 14/06/2017

pelo que

$$\operatorname{cis}(\theta)+\operatorname{cis}(-\theta)=\cos(\theta)+i\sin(\theta)+\cos(\theta)-i\sin(\theta)=2\cos(\theta)$$

Logo

$$\cos(\theta)=\frac{\operatorname{cis}(\theta)+\operatorname{cis}(-\theta)}{2}. \tag{1}$$

b. Seja $z=\operatorname{cis}(\theta)$. Então

$$\overline{z}=\overline{\cos(\theta)+i\sin(\theta)}=\cos(\theta)-i\sin(\theta)=\cos(-\theta)+i\sin(-\theta)=\operatorname{cis}(-\theta). \quad (*)$$

Pretendemos calcular

$$\cos^6(\theta)=\left(\frac{\operatorname{cis}(\theta)+\operatorname{cis}(-\theta)}{2}\right)^6=\frac{1}{64}\left(z+\overline{z}\right)^6.$$

Usando o binómio de Newton

$$(1+x)^n=\sum_{k=0}^{n}{}^nC_k\,x^k \tag{2}$$

vem, para $y\neq 0$

$$(x+y)^n=y^n\left(1+\frac{x}{y}\right)^n=y^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{y}\right)^k=\sum_{k=0}^{n}{}^nC_k\,x^ky^{n-k}. \tag{3}$$

Substituindo x e y por, respectivamente, z e $\overline{z}$, tem-se

$$\begin{aligned}(z+\overline{z})^6&=\sum_{k=0}^{6}{}^6C_k\,z^k\overline{z}^{6-k}\\&=z^6+6z^5\overline{z}+15z^4\overline{z}^2+20z^3\overline{z}^3+15z^2\overline{z}^4+6z\overline{z}^5+\overline{z}^6,\end{aligned}$$

dado que ${}^6C_0={}^6C_6=1, {}^6C_1={}^6C_5=6, {}^6C_2={}^6C_4=15$ e ${}^6C_3=20$. Então

$$\begin{aligned}\cos^6(\theta)&=\frac{1}{64}\left(z^6+6z^5\overline{z}+15z^4\overline{z}^2+20z^3\overline{z}^3+15z^2\overline{z}^4+6z\overline{z}^5+\overline{z}^6\right)\\&=\frac{1}{64}\left(z^6+6z^4\left(z\overline{z}\right)+15z^2\left(z\overline{z}\right)^2+20\left(z\overline{z}\right)^3\right)\\&+\frac{1}{64}\left(15\overline{z}^2\left(z\overline{z}\right)^2+6\overline{z}^4\left(z\overline{z}\right)+\overline{z}^6\right)\\&=\frac{1}{64}\left(z^6+6z^4+15z^2+20+15\overline{z}^2+6\overline{z}^4+\overline{z}^6\right)\end{aligned}$$

visto que

$$z\overline{z} = 1 = (z\overline{z})^2 = z^2\overline{z}^2 = (z\overline{z})^3 = z^3\overline{z}^3. \tag{**}$$

Pela fórmulas de Moivre, para $n \in \mathbb{N}$, tem-se

$$\begin{aligned}
z^n &= (\mathrm{cis}(\theta))^n = \mathrm{cis}(n\theta) = \cos(n\theta) + i\sin(n\theta)\\
\overline{z}^n &= (\mathrm{cis}(-\theta))^n = \mathrm{cis}(-n\theta) = \cos(n\theta) - i\sin(n\theta)\\
z^n + \overline{z}^n &= 2\cos(n\theta) \qquad (4)
\end{aligned}$$

Assim,

$$\begin{aligned}
z^6 + \overline{z}^6 &= 2\cos(6\theta)\\
z^4 + \overline{z}^4 &= 2\cos(4\theta)\\
z^2 + \overline{z}^2 &= 2\cos(2\theta)
\end{aligned}$$

vindo, finalmente

$$\begin{aligned}
\cos^6(\theta) &= \frac{1}{64}\left((z^6 + \overline{z}^6) + 6\left(z^4 + \overline{z}^4\right) + 15\left(z^2 + \overline{z}^2\right) + 20\right)\\
&= \frac{1}{32}\left(\cos(6\theta) + 6\cos(4\theta) + 15\cos(2\theta) + 10\right)\\
&= \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}. \qquad (5)
\end{aligned}$$

c. Calculando $z - \overline{z}$

$$\begin{aligned}
z - \overline{z} &= \mathrm{cis}(\theta) - \mathrm{cis}(-\theta) = \cos(\theta) + i\sin(\theta) - ((\cos(-\theta) + i\sin(-\theta))\\
&= \cos(\theta) + i\sin(\theta) - \cos(\theta) + i\sin(\theta)\\
&= 2i\sin(\theta) \qquad (***)
\end{aligned}$$

obtém-se

$$\sin\theta = \frac{z - \overline{z}}{2i} = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{6}$$

Assim

$$\sin^4(\theta) = \left(\frac{z - \overline{z}}{2i}\right)^4 = \frac{(z - \overline{z})^4}{16}$$

Substituindo y por $-y$ em (3) obtém-se

$$(x - y)^n = \sum_{k=0}^{n} {}^nC_k\, x^k\, (-y)^{n-k} = \sum_{k=0}^{n} (-1)^{n-k}\ {}^nC_k\, x^k y^{n-k}. \tag{7}$$

Aplicando a $(z-\overline{z})^4$, vem, sucessivamente, atendendo a que ${}^4C_0 = {}^4C_4 = 1, {}^4C_1 = {}^4C_3 = 4$, e ${}^4C_2 = 6$:

$$\begin{aligned}
(z-\overline{z})^4 &= \sum_{k=0}^{4}(-1)^{4-k}\,{}^4C_k\, z^k\overline{z}^{4-k} \\
&= {}^4C_0\,\overline{z}^4 - {}^4C_1\, z\overline{z}^3 + {}^4C_2\, z^2\overline{z}^2 - {}^4C_3\, z^3\overline{z} + {}^4C_4\, z^4 \\
&= \overline{z}^4 - 4\, z\overline{z}^3 + 6\, z^2\overline{z}^2 - 4\, z^3\overline{z} + z^4 \\
&= \overline{z}^4 + z^4 - 4\,\overline{z}^2 + 6 - 4\, z^2 \\
&= (z^4+\overline{z}^4) - 4(z^2+\overline{z}^2) + 6 \\
&= 2\cos(4\theta) - 8\cos(2\theta) + 6
\end{aligned}$$

justificando-se assim que

$$\sin^4(\theta) = \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}. \tag{8}$$

Quanto a $\sin\left(\frac{\pi}{12}\right)$, calculamos primeiro $\sin^4(\frac{\pi}{12})$, fazendo $\theta = \pi/12$ na fórmula (8):

$$\begin{aligned}
\sin^4\left(\frac{\pi}{12}\right) &= \frac{1}{8}\cos\left(4\times\frac{\pi}{12}\right) - \frac{1}{2}\cos\left(2\times\frac{\pi}{12}\right) + \frac{3}{8} \\
&= \frac{1}{8}\cos\left(\frac{\pi}{3}\right) - \frac{1}{2}\cos\left(\frac{\pi}{6}\right) + \frac{3}{8} \\
&= \frac{1}{8}\times\frac{1}{2} - \frac{1}{2}\times\frac{\sqrt{3}}{2} + \frac{3}{8} \\
&= \frac{7-4\sqrt{3}}{16}.
\end{aligned}$$

Por isso,

$$\sin\left(\frac{\pi}{12}\right) = \sqrt[4]{\sin^4\left(\frac{\pi}{12}\right)} = \sqrt[4]{\frac{7-4\sqrt{3}}{16}} = \frac{\sqrt[4]{7-4\sqrt{3}}}{2}. \tag{9}$$

Queluz, 14/06/2017
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